Ehrenfeucht-Fraïssé Games and Hanf's Locality Theorem

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- The number of rounds, $q \in \mathbb{N}$, is fixed in advance.

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Each round has three steps.

Steps in a Round

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2. The **spoiler** chooses an element of that structure.
3. The **duplicator** chooses an element of the other structure.

Winning Condition

After q rounds, we have a pair (a_i, b_i) for each $i \in \{1, \ldots, q\}$, where a_i and b_i are the elements chosen in round i in $\mathfrak{A}$ and $\mathfrak{B}$, respectively.

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Otherwise, the **spoiler** wins.

Examples of Games on Strict Linear Orders

- $\mathfrak{A} = ([7], <)$ and $\mathfrak{B} = ([9], <)$.
- $\mathfrak{A} = (\mathbb{Z}, <)$ and $\mathfrak{B} = (\mathbb{Q}, <)$.
- $\mathfrak{A} = (\mathbb{Q}, <)$ and $\mathfrak{B} = (\mathbb{R}, <)$.

The Ehrenfeucht–Fraïssé Theorem

Theorem (Ehrenfeucht–Fraïssé)

Let $\mathcal{L}$ be a first-order relational language, let $\mathfrak{A}$ and $\mathfrak{B}$ be two $\mathcal{L}$-structures, and let $q \in \mathbb{N}$. The **duplicator** has a winning strategy in the q-round Ehrenfeucht–Fraïssé game on $\mathfrak{A}$ and $\mathfrak{B}$ if and only if $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same first-order sentences of quantifier rank at most q.

Some Definitions

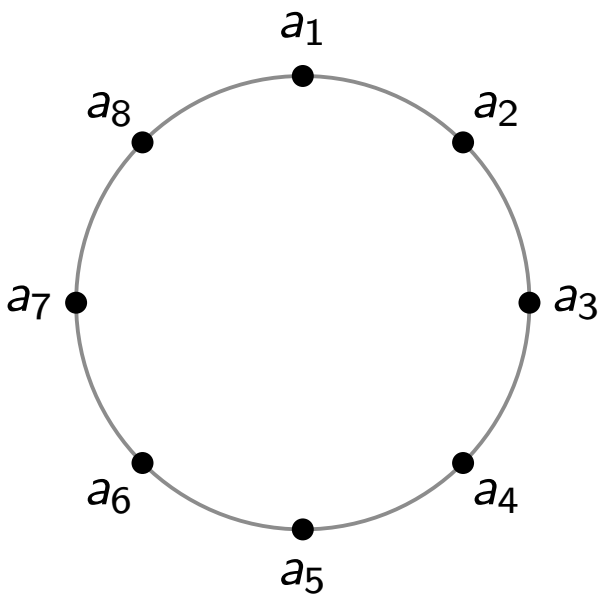
Every relational structure has an associated graph, called its **Gaifman graph**. Two distinct elements are adjacent if they occur together in a tuple of some relation.

For $r \in \mathbb{N}$, the r-**neighborhood** of an element a of $\mathfrak{A}$, denoted by $\mathcal{N}_r^{\mathfrak{A}}(a)$, is the substructure induced by the ball of radius r around a in the Gaifman graph, with a as its distinguished center.

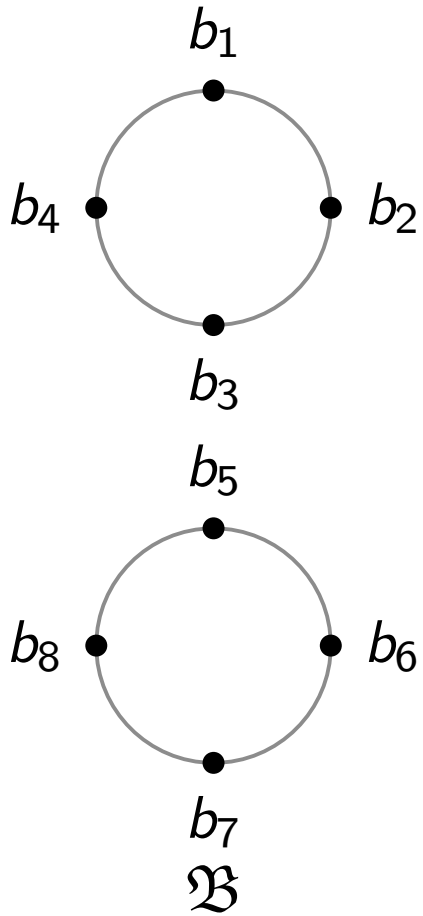
Definition

Two structures $\mathfrak{A}$ and $\mathfrak{B}$ are **locally isomorphic at radius** r, written $\mathfrak{A} \cong_r \mathfrak{B}$, if there is a bijection $f : A \to B$ such that, for every $a \in A$, the pointed r-neighborhoods of a and $f(a)$ are isomorphic.

Example of Local Isomorphism at Radius 1

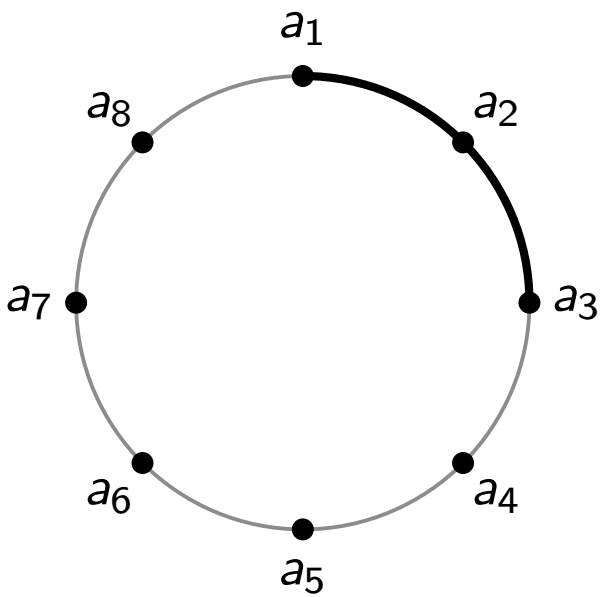


$\mathfrak{A}$

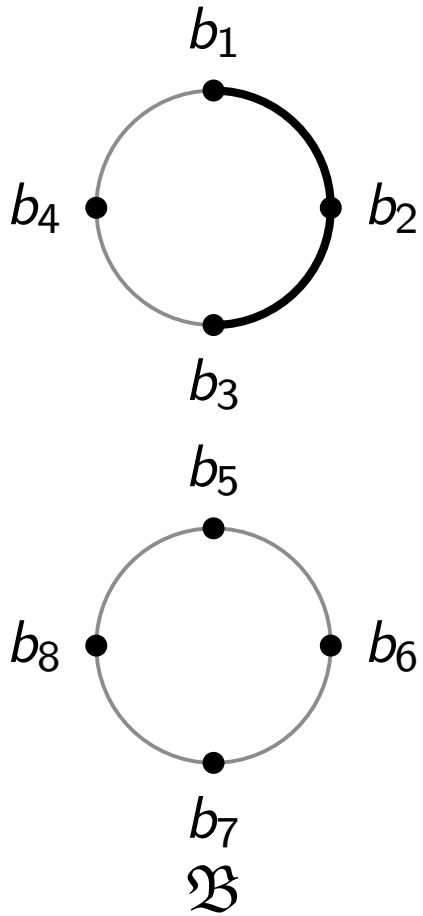


$\mathfrak{B}$

Example of Local Isomorphism at Radius 1



$\mathfrak{A}$



$\mathfrak{B}$

Hanf's Theorem

Theorem (Hanf)

Let $\mathcal{L}$ be a first-order relational language. For every $q \in \mathbb{N}$, there is a radius $r(q) \in \mathbb{N}$ such that, for any two locally finite $\mathcal{L}$-structures $\mathfrak{A}$ and $\mathfrak{B}$, if $\mathfrak{A} \cong_{r(q)} \mathfrak{B}$, then $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same first-order sentences of quantifier rank at most q.

Duplicator's strategy : After each round k of the q-round game, maintain an isomorphism

$$\bigcup_{i=1}^{k} \mathcal{N}_{3^{q-k}}^{\mathfrak{A}}(a_i) \cong \bigcup_{i=1}^{k} \mathcal{N}_{3^{q-k}}^{\mathfrak{B}}(b_i)$$

that sends each a_i to b_i.

What Does This Tell Us?

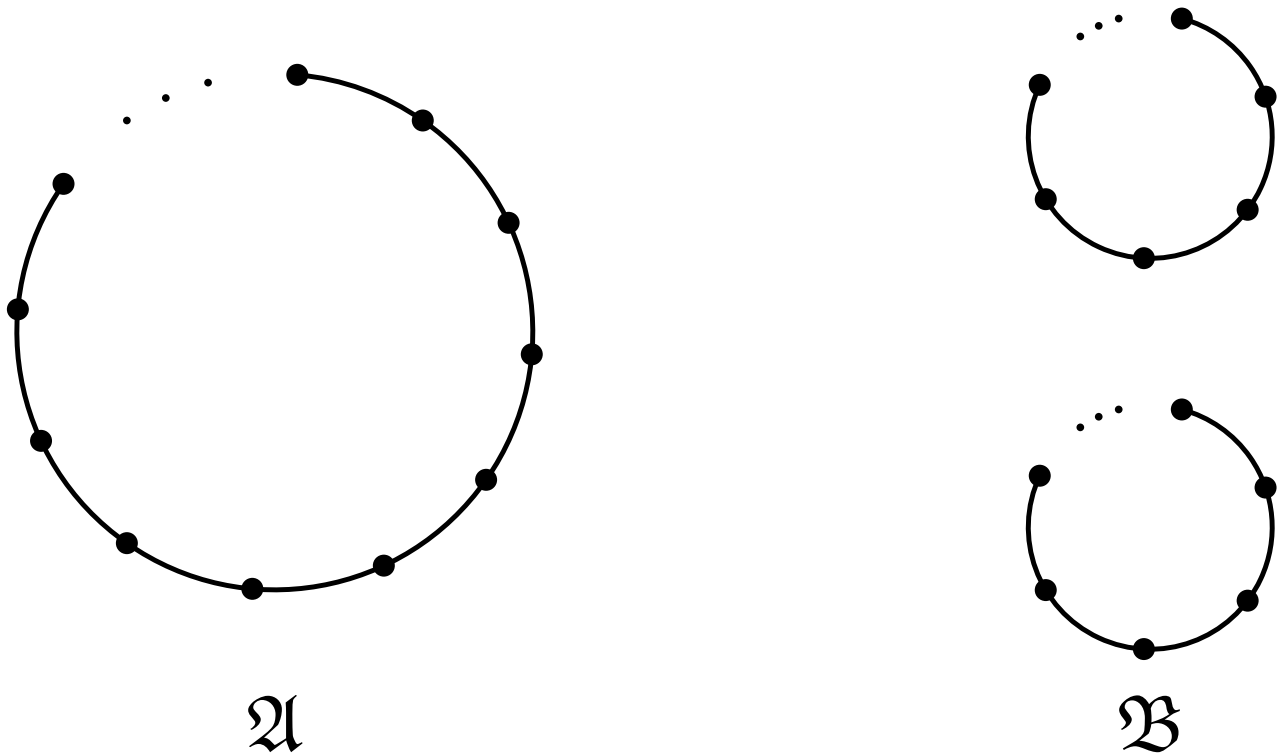
We can use Hanf's theorem to show that a property P cannot be expressed by a first-order sentence.

Here are some limits of the expressive power of first-order logic.

- No first-order sentence expresses the property of being a connected graph.
- Acyclicity, planarity, and Hamiltonicity are also inexpressible in first-order logic, as is k-colorability for each fixed $k \geq 2$.
- The property of having even, odd, finite or infinite cardinality are also inexpressible by a first-order sentence.

Connectivity of a graph

Suppose there is a first-order sentence Φ that expresses graph connectivity. Let q be the quantifier rank of Φ. By Hanf's locality theorem, there is a radius $r = r(q)$ such that, if $\mathfrak{A} \cong_r \mathfrak{B}$, then $\mathfrak{A} \models \Phi \iff \mathfrak{B} \models \Phi$.



$\mathfrak{A}$ $\mathfrak{B}$

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